1. *Find the formula for the solution $x(t)$ of the problem*

$$x' + \frac{x}{1+t} = 1, \qquad x(0) = 0\,,$$

in the interval $(0, \infty)$.

Solution:
There are many ways to do this problem. One can use for example formula (2.14) in the textbook (or formula (23) in the lecture log). One can also proceed just by the rule "first solve the homogeneous equation and then do the variation of constants". Here we will do the calculation by using the last mentioned procedure. The homogeneous equation $x' + \frac{x}{1+t} = 0$ can be solved by integrating $\frac{dx}{x} = -\frac{dt}{1+t}$ ("separation of variables"). The general solution of the homogeneous equation is $x(t) = \frac{C}{1+t}$, and hence we should seek the solution the inhomogeneous equation as $x(t) = \frac{C(t)}{1+t}$. Substituting the expression into the equation and using the initial condition $x(0) = 0$, we obtain $C(t) = \int_0^t (1+t)dt = t + \frac{t^2}{2}$. Hence $x(t) = \frac{t+\frac{1}{2}t^2}{1+t}$.
Several students noticed that there is a quick way to solve the equation by re-writing it as $[(1+t)x]' = 1+t$. Integration of this identity between 0 and t together with $x(0) = 0$ then gives $(1+t)x(t) = t + \frac{1}{2}t^2$, which for $t \in (0, \infty)$ is of course the same as $x(t) = \frac{t+\frac{1}{2}t^2}{1+t}$.

2. *For $a \le 1$ find the formula for the solution $x(t)$ of the problem*

$$x' = \frac{x^2}{t^2}, \qquad x(1) = a\,,$$

in the interval $(1, \infty)$.

Solution:
We write the equation as $\frac{dx}{x^2} = \frac{dt}{t^2}$ and integrate the left-hand-side between a and x $[= x(t)]$ and the right-hand side between 1 and t. We obtain $-\frac{1}{x} + \frac{1}{a} = -\frac{1}{t} + 1$. Therefore $x = x(t) = \frac{1}{\frac{1}{t}+\frac{1}{a}-1} = \frac{at}{a+t(1-a)}$. (There are many other ways in which the formula can be written,of course.) Instead of using definite integrals ($\int_a^x \frac{d\tilde{x}}{\tilde{x}^2} = \int_1^t \frac{d\tilde{t}}{\tilde{t}^2}$) one can use indefinite integrals and write $\int \frac{dx}{x^2} = \int \frac{dt}{t^2} + C$, which after integration gives $-\frac{1}{x} = -\frac{1}{t} + C$. The value of C is then calculated from the initial condition: at $t = 1$ we must have $-\frac{1}{a} = -\frac{1}{1} + C$, which gives $C = 1 - \frac{1}{a}$, leading of course to the same formula for $x = x(t)$ as above.

3. *A point moves on a smooth horizontal surface. Assume the motion is along a straight line and is governed by the equation*

$$x'' + \gamma x' = 0\,,$$

where $\gamma > 0$ is a coefficient of friction and $x = x(t)$ is the coordinate of the point in some natural coordinate system on the line. For concreteness we can assume that x is measured in meters and the time is measured in seconds. Assume that

$$x'(0) = v > 0\,.$$

Show that – according to our equation – the motion of the point will never stop, but the distance

$$d = \lim_{t\to\infty} \left(x(t) - x(0) \right)$$

is always finite. Determine γ if we know that for $v = 1$ [meter per second] we have $d = 10$ [meters].

Solution:

We are dealing with a second order equation with constant coefficients. The characteristic equation is $\lambda^2 + \gamma\lambda = 0$, with the roots $\lambda_1 = 0$ and $\lambda_2 = -\gamma$. The general solution therefore is $x(t) = C_1 e^{0\cdot t} + C_2 e^{-\gamma t} = C_1 + C_2 e^{-\gamma t}$. The initial condition $x'(0) = v$ implies $C_2 = -\frac{v}{\gamma}$ and we see that $x(t) - x(0) = \frac{v}{\gamma}(1 - e^{-\gamma t})$. The velocity $x'(t) = ve^{-\gamma t}$ cannot vanish for any $t > 0$ as $v > 0$ by our assumptions, and we also see that the limit $d = \lim_{t\to\infty}(x(t) - x(0))$ exists, is finite and, in fact, $d = \frac{v}{\gamma}$. Therefore $\gamma = \frac{v}{d}$. For our specific values of v and d we obtain $\gamma = \frac{1}{10}\ \sec^{-1}$.

4. *Consider the 2×2 system of equations*

$$\begin{aligned} \dot{x}_1 &= -x_1 + ax_2\,, \\ \dot{x}_2 &= x_1 - x_2\,, \end{aligned}$$

where $a \in \mathbf{R}$ is a parameter. Find all values of a for which the the system has both of the following two properties:

- *All solutions on the interval $(0, \infty)$ approach a finite limit as $t \to \infty$. In other words, for $j = 1, 2$ the limit $\lim_{t\to\infty} x_j(t)$ exists and is finite.*
- *There exists at least one solution in $(0, \infty)$ for which the quantity $x_1^2(t) + x_2^2(t)$ does not approach 0 as $t \to \infty$.*

Solution:

We are dealing with a 2×2 system of the form $x' = Ax$ with $A = \begin{pmatrix} -1 & a \\ 1 & -1 \end{pmatrix}$. The characteristic polynomial of A is $\det(A - \lambda I) = (-1-\lambda)^2 - a = (1+\lambda)^2 - a$. The eigenvalues are found from the equation $(1+\lambda)^2 - a = 0$, which is the same as $\lambda = -1 \pm \sqrt{a}$. Let us denote $\lambda_1 = -1 - \sqrt{a}$ and $\lambda_2 = -1 + \sqrt{a}$. If $a < 0$, the eigenvalues will be complex. Let us now consider two cases:

Case 1: $a \neq 0$. In this case we have $\lambda_1 \neq \lambda_2$ and hence there exists a basis of $\mathbf{C}^2$ consisting of eigenvectors. Let $x^{(1)}$, $x^{(2)}$ be such a basis, with $Ax^{(k)} =$

$\lambda_k x^{(k)}$, $k = 1, 2$. (We do not have to calculate $x^{(1)}$, $x^{(2)}$ explicitly to solve this problem.) The general solution of our system is $x(t) = C_1 x^{(1)} e^{\lambda_1 t} + C_2 x^{(2)} e^{\lambda_2 t}$. Denoting by $\Re z$ the real part of a (possibly complex) number z, we note that $\Re\lambda_1 \leq -1$ and hence $\lim_{t\to\infty} C_1 x^{(1)} e^{\lambda_1 t} = 0$ for any a. For $a < 1$ we also have $\Re\lambda_2 < 0$, and hence $\lim_{t\to\infty} x(t) = 0$ for any solution. Hence no $a < 1,\ a \neq 0$ will satisfy the second condition required in the formulation of the problem. We have ruled out all $a < 1, a \neq 0$ as solutions.
For $a > 1$ we have $\lambda_2 > 0$, and for $C_2 \neq 0$ we have $\lim_{t\to\infty} |C_2 x^{(2)} e^{\lambda_2 t}| = +\infty$. Hence for $a > 1$ there always are solutions of the system which will not satisfy the first condition in the formulation of the problem. We have ruled out all $a > 1$ as solutions.
When $a = 1$, we have $\lambda_1 = -2$, $\lambda_2 = 0$ and the general solution is $x(t) = C_1 x^{(1)} e^{-2t} + C_2 x^{(2)}$. These functions have a finite limit $C_2 x^{(2)}$ as $t \to \infty$. Moreover, the limit is non-zero if $C_2 \neq 0$. Hence for $a = 1$ the system will satisfy both requirements in the formulation of the problem.

Case 2: $a = 0$. In this case we have a double eigenvalue $\lambda = -1$. As the eigenvalue is strictly negative, it is reasonable to expect that all solutions will converge to 0 as $t \to \infty$.[1] One can verify this in several ways. If we do not wish to use the matrix exponentials or other "general principles", we can proceed with a verification "by hand".[2] Note that, as we now assume $a = 0$, the first equation of the system is $\dot{x}_1 = -x_1$, with the general solution $x_1(t) = C_1 e^{-t}$. The second equation then is $\dot{x}_2 = -x_2 + C_1 e^{-t}$. The general solution of the homogeneous equation $\dot{x}_2 = -x_2$ is $c_2 e^{-t}$, and we seek the solution of the inhomogeneous equation as $x_2(t) = c_2(t) e^{-t}$. This gives $c_2'(t) = C_1$, and therefore $c_2(t) = C_1 t + C_2$. Thus the general solution of the system is given by $x_1(t) = C_1 e^{-t}, \quad x_2(t) = C_1 t e^{-t} + C_2 e^{-t}$. All these solutions converge to 0 as $t \to \infty$, and hence $a = 0$ is ruled out as a solution to our problem.
Putting together the conclusions from both Case 1 and Case 2, we see that $a = 1$ is the only solution to our problem.

Instead of using the calculation "by hand" we can use the matrix exponentials. The general solution of the system can be written $x(t) = e^{tA} x^{(0)}$, where $x^{(0)} \in \mathbf{C}^2$. For $a = 0$ our matrix is of the from $-I + J$, with $J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Note that $J^2 = 0$ and hence $e^{tJ} = I + tJ$, which gives $e^{tA} = e^{t(-I+J)} = e^{-tI} e^{tJ} = \begin{pmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{pmatrix}$.

For $a \neq 0$ we have $e^{tA} = P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} P^{-1}$ for a suitable regular matrix P and consideration similar to those above again show that $a = 1$ is the only solution to our problem.

[1] One can in fact prove that for any $n \times n$ matrix A for which all the eigenvalues are strictly negative (multiple eigenvalues are allowed) all solutions of $\dot{x} = Ax$ converge to 0 for $t \to \infty$ with an exponential rate. The proof is an easy application of the Jordan canonical form.

[2] We did such a calculation in lecture 17, see the lecture log, system (309) on page 60.